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Nonlinear Electrohydrodynamic Instability Conditions of an Interface between Two Fluids under the Effect of a Normal Periodic Electric Field. III

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A charge-free surface separating two semi-infinite dielectric fluids influenced by a normal periodic electric field is subjected to nonlinear deformations. We use the method of multiple scales in order to solve the nonlinear equations. In the first-order problem we obtained Mathieu's differential equation. For the second order, we obtain the nonhomogeneous Mathieu equation and we use the method of multiple scales to obtain a sequence of equations. In the third order we obtain the second-order differential equation of periodic coefficients. Also, we obtain a formula for surface elevation. The stability conditions are determined.

1. INTRODUCTION

Electrohydrodynamics can be regarded as a branch of fluid mechanics concerned with electric force effects. It can also be considered as that part of electrodynamics which is involved with the influence of moving media on electric fields.

Very few studies on nonlinear electrohydrodynamic Rayleigh–Taylor instability have been attempted.

Melcher (1963) and Michael (1977) studied the nonlinear stability of the interface of a fluid of finite depth stressed by a normal electric field. In their models, there are charges on the interface. They studied conducting fluids, and therefore the effect of the dielectric constants is not accounted for in their analysis. The nonlinear cutoff wavenumbers were not evaluated in their studies.

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Kant *et al.* (1981) investigated the stability of weakly nonlinear waves on the surface of a fluid layer in the presence of an applied electric field by using the derivative expansion method. They also studied conducting fluids, and therefore the effect of the dielectric constants was also not accounted for in their analysis.

The nonlinear electrohydrodynamic Rayleigh–Taylor instability was investigated by Mohamed and El Shehawey (1983*a*,*b*; 1984). They studied a charge-free surface separating two semi-infinite dielectric fluids influenced by a normal electric field and a tangential electric field subjected to nonlinear deformation. They obtained two nonlinear Schrödinger equations by means of which one can deduce the cutoff wavenumber and analyze the stability of the system.

The method of multiple-scale perturbations was used successfully by Hasimoto and Ono (1972) for fluids of finite depth and by Nayfeh (1976) for fluids of infinite depth.

In the 1960s a new and wide field of periodic flow phenomena was discovered in the earth's atmosphere by meteorological satellites: the Van Karman streetlike vortex trails behind the island of Madeira, Jan Mayen, and Guadalupe (Baja California), to name only the three most significant locations (Berger and Wille, 1972).

A real-valued function of one variable $F(x)$ defined for all real x is said to be periodic and to have period p if, and only if, for fixed $p>0$ and for all x

$$F(x+p)=F(x)$$

Note that if $F(x)$ is periodic with period p, then it is also periodic with period Kp, $K=2, 3, \ldots, n-1, n$, and, moreover, it is possible, but not necessary, that $F(x)$ is periodic with period p^*, where $0<p^*<p$ (Rouch and Mawhin, 1978).

In this paper we study the effect of a time-dependent normal electric field on the stability of a dielectric liquid in the absence of surface charges. The method of multiple-scale perturbations has been used to obtain information about solutions of equations that involve difficulties, such as equations with variable coefficients, doubly periodic functions, and for singular perturbed systems, etc. (Nayfeh, 1973, 1977; Nayfeh and Mook, 1977, 1979).

2. STATEMENT OF THE PROBLEM

In this section, we consider two semi-infinite dielectric inviscid fluids separated by the plane $y=0$. The upper and lower densities of the fluids are $\rho^{(2)}$ and $\rho^{(1)}$, respectively. Both of the fluids are subject to a periodic electric field in the y direction ($E_0^{(2)} \cos \omega_0 T_0$ and $E_0^{(1)} \cos \omega_0 T_0$, respectively). We shall assume that there are no surface charges at the surface of separation in the equilibrium state, and therefore the electric displacement is continuous at the interface. As the motion of the system starts from rest, it is taken to be an irrotational flow. The motion is governed by the following equations:

$$\nabla^2 \phi^{(2),(1)}(x, y, t) = 0 \tag{2.1}$$

where $\phi(x, y, t)$ is the velocity potential,

$$V^{(2),(1)} = \nabla \phi^{(2),(1)}$$

We shall assume that the quasistatic approximation is valid and we introduce the electrostatic potential $\psi^{(2),(1)}$ such that

$$\mathbf{E}^{(2),(1)} = \mathbf{E}_0^{(2),(1)} \cos \omega_0 t \, \mathbf{e}_y - \nabla \psi^{(2),(1)} \tag{2.2}$$

Therefore the differential equation satisfied by $\psi^{(2),(1)}$ is the Laplace equation

$$\frac{\partial^2 \psi^{(2),(1)}}{\partial x^2} + \frac{\partial^2 \psi^{(2),(1)}}{\partial y^2} = 0 \tag{2.3}$$

The superscripts (1) and (2) refer to quantities in the lower fluid and upper fluid, respectively.

In our analysis the various quantities are nondimensionalized using the characteristic length $L = (T/\rho^{(1)} g')^{1/2}$ and the characteristic time $(L/g')^{1/2}$, where T is the surface tension and g' is the acceleration due to gravity acting in the negative y direction.

2.1. Boundary Conditions

(i) The kinematic boundary condition is

$$\eta_t + \eta_x \phi_x = \phi_y \qquad \text{at} \quad y = \eta(x, t) \tag{2.4}$$

(ii) The tangential component of electric field should be continuous at the interface on $y = \eta(x, t)$,

$$\eta_x + [\psi_y] + [\psi_x] = \eta_x \cos \omega_0 T_0 [E_0] \qquad \text{at} \quad y = \eta(x, t) \tag{2.5}$$

where $[\cdot]$ represents the jump across the interface.

(iii) Since there are no surface charges at the surface $y = \eta(x, t)$, the normal electric displacement is continuous at the interface,

$$\eta_x [\tilde{\varepsilon} \psi_x] = [\tilde{\varepsilon} \psi_y] \qquad \text{at} \quad y = \eta(x, t) \tag{2.6}$$

(iv) The stress tensor is given by

$$\Pi_{ij} = -\Pi \delta_{ij} + \tilde{\varepsilon} E_i E_j - \tfrac{1}{2} \tilde{\varepsilon} E_\nu^2 \delta_{ij}, \tag{2.7}$$

where $\Pi = P - \frac{1}{2} \tilde{\varepsilon} E^2$.

The normal hydrodynamic stress is balanced by the normal electric stress. The balance condition is then

$$\begin{aligned}
&\phi_t^{(1)}-\rho\phi_t^{(2)}+\tfrac{1}{2}[(\phi_x^{(1)})^2-\rho(\phi_x^{(2)})^2]+\tfrac{1}{2}[(\phi_y^{(1)})^2-\rho(\phi_y^{(2)})^2]+(1-\rho)\eta \\
&\quad=\eta_{xx}(1+\eta_x^2)^{-3/2}-\tfrac{1}{2}[\tilde{\varepsilon}\psi_x^2]+\tfrac{1}{2}[\tilde{\varepsilon}\psi_y^2]-[\tilde{\varepsilon}E_0\psi_y]\cos\omega_0T_0 \\
&\qquad-\eta_x^2[\tilde{\varepsilon}E_0^2]\cos^2\omega_0T_0+\eta_x^2[\tilde{\varepsilon}\psi_x^2]-\eta_x^2[\tilde{\varepsilon}\psi_y^2] \\
&\qquad+2\eta_x^2[\tilde{\varepsilon}E_0\psi_y]\cos\omega_0T_0+2\eta_x[\tilde{\varepsilon}\psi_xE_0]\cos\omega_0T_0-2\eta_x^3 \\
&\qquad\times[\tilde{\varepsilon}E_0\psi_x]\cos\omega_0T_0-2\eta_x[\tilde{\varepsilon}\psi_x\psi_y]+2\eta_x^3[\tilde{\varepsilon}\psi_x\psi_y] \\
&\qquad\quad\text{at}\quad y=\eta(x,t)
\end{aligned}\tag{2.8}$$

2.2. Method of Solution and Analysis

The set of equations (2.1), (2.3), (2.4)-(2.6), and (2.8) will be solved using the method of multiple scales (Nayfeh, 1973, 1976). We expand the various variables in ascending powers in terms of a small dimensionless parameter ε characterizing the steepness ratio of the wave. The independent variables x, t are scaled in a like manner,

$$X_n=\varepsilon^n x,\qquad T_n=\varepsilon^n t \tag{2.9}$$

and the variables may be expanded as

$$\eta(x,t)=\sum_{n=1}^{3}\varepsilon^n\eta_n(X_0,X_1,X_2;\,T_0,T_1,T_2)+O(\varepsilon^4) \tag{2.10}$$

$$\psi^{(2),(1)}(x,y,t)=\sum_{n=1}^{3}\varepsilon^n\psi_n^{(2),(1)}(X_0,X_1,X_2;y,T_0,T_1,T_2)+O(\varepsilon^4) \tag{2.11}$$

$$\phi^{(2),(1)}(x,y,t)=\sum_{n=1}^{3}\varepsilon^n\psi_n^{(2),(1)}(X_0,X_1,X_2;y,T_0,T_1,T_2)+O(\varepsilon^4) \tag{2.12}$$

Substituting from (2.9)-(2.11) and (2.12) into (2.1), (2.3), (2.4)-(2.6), and (2.8) and equating the coefficients of the respective powers of ε, the following three orders of the problems are obtained.

Order ε:

$$\frac{\partial^2\psi_1^{(2),(1)}}{\partial X_0^2}+\frac{\partial^2\psi_1^{(2),(1)}}{\partial y^2}=0 \tag{2.13}$$

$$\frac{\partial^2\phi_1^{(2),(1)}}{\partial X_0^2}+\frac{\partial^2\phi_1^{(2),(1)}}{\partial y^2}=0 \tag{2.14}$$

$$\frac{\partial \eta_1}{\partial T_0}-\frac{\partial \phi_1^{(2),(1)}}{\partial y}=0 \quad \text{at} \quad y=0 \tag{2.15}$$

$$\left[\frac{\partial \psi_1}{\partial X_0}\right]=\frac{\partial \eta_1}{\partial X_0}[E_0]\cos\omega_0 T_0 \quad \text{at} \quad y=0 \tag{2.16}$$

$$\left[\tilde{\varepsilon}\frac{\partial \psi_1}{\partial y}\right]=0 \quad \text{at} \quad y=0 \tag{2.17}$$

$$\frac{\partial \phi_1^{(1)}}{\partial T_0}-\rho\frac{\partial \phi_1^{(2)}}{\partial T_0}+(1-\rho)\eta_1-\frac{\partial^2 \eta_1}{\partial X_0^2}+\left[\tilde{\varepsilon}E_0\frac{\partial \psi_1}{\partial y}\right]\cos\omega_0 T_0=0 \quad \text{at} \quad y=0 \tag{2.18}$$

The solutions of these equations can be written in the form

$$\eta_1(X_0, X_1, X_2;\, T_0, T_1, T_2)=D(X_1, X_2;\, T_0, T_1, T_2)\, e^{iKX_0}+\text{c.c.} \tag{2.19}$$

$$\phi_1^{(1)}(X_0, X_1, X_2, y;\, T_0, T_1, T_2)=\frac{1}{K}\frac{\partial D}{\partial T_0}\, e^{iKX_0+Ky}+\text{c.c.} \tag{2.20}$$

$$\phi_1^{(2)}(X_0, X_1, X_2, y;\, T_0, T_1, T_2)=-\frac{1}{K}\frac{\partial D}{\partial T_0}\, e^{iKX_0-Ky}+\text{c.c.} \tag{2.21}$$

$$\psi_1^{(1)}(X_0, X_1, X_2, y;\, T_0, T_1, T_2)=D\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}\cos\omega_0 T_0\, e^{iKX_0+Ky}+\text{c.c.} \tag{2.22}$$

$$\psi_1^{(2)}(X_0, X_1, X_2, y;\, T_0, T_1, T_2)=-D\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(2)}\cos\omega_0 T_0\, e^{iKX_0-Ky}+\text{c.c.} \tag{2.23}$$

where c.c. denotes complex conjugate.

Substituting from equations (2.19)-(2.22) and (2.23) into equation (2.18) and after some simplifications, we obtain the following differential equation for $D(T_0)$, since $D(T_0)$ represents the first-order amplitude of the deformation at the interface:

$$\frac{\partial^2 D}{\partial T_0^2}+\frac{1}{1+\rho}\left\{1-\rho+K^2-K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 T_0\right\}D=0 \tag{2.24}$$

Let us use the following notations for simplicity:

$$a=\frac{K}{\omega_0^2(1+\rho)}\left(1-\rho+K^2-\frac{K}{2}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\right)$$

$$q=\frac{K^2}{4\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}$$

Then equation (2.24) can be written as

$$\frac{\partial^2 D}{\partial \xi^2}+(a-2q\cos 2\xi)D=0 \tag{2.25}$$

where

$$\xi=\omega_0 T_0$$

Equation (2.25) is the Mathieu differential equation; we assume a regular perturbation expansion for D:

$$D=D_0+\varepsilon_1 D_1+\varepsilon_1^2 D_2+O(\varepsilon_1^3) \tag{2.26}$$

where $\varepsilon_1=q$.

Substituting equation (2.26) into equation (2.25) and comparing like powers of ε_1 gives a sequence of equations, and using the perturbation theory, we solve all orders. After solving to all orders in perturbation theory we get (Bender and Orszag, 1978)

$$D(X_1,X_2,T_0,T_1,T_2)=A(X_1,X_2,T_0,T_1,T_2)\,e^{i\sqrt{a}\omega_0 T_0}+\text{c.c.} \tag{2.27}$$

where

$$A(X_1,X_2,T_0,T_1,T_2)=\sum \varepsilon_1^n A_n(X_1,X_2,T_1,T_2)\,e^{2in\omega_0 T_0}$$

Order ε^2:

$$\frac{\partial^2 \phi_2^{(2),(1)}}{\partial X_0^2}+\frac{\partial^2 \phi_2^{(2),(1)}}{\partial y^2}=-2\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0\,\partial X_1} \tag{2.28}$$

$$\frac{\partial^2 \psi_2^{(2),(1)}}{\partial X_0^2}+\frac{\partial^2 \psi_2^{(2),(1)}}{\partial y^2}=-2\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_0\,\partial X_1} \tag{2.29}$$

$$\frac{\partial \eta_2}{\partial T_0}-\frac{\partial \phi_2^{(2),(1)}}{\partial y}=\eta_1\frac{\partial^2 \phi_1^{(2),(1)}}{\partial y^2}-\frac{\partial \eta_1}{\partial T_1}-\frac{\partial \eta_1}{\partial X_0}\frac{\partial \phi_1^{(2),(1)}}{\partial X_0} \quad \text{at} \quad y=0 \tag{2.30}$$

$$2\frac{\partial \eta_1}{\partial X_0}\left[\frac{\partial \psi_1}{\partial y}\right]+\left[\frac{\partial \psi_2}{\partial X_0}\right]+\eta_1\left[\frac{\partial^2 \psi_1}{\partial y\,\partial X_0}\right]+\left[\frac{\partial \psi_1}{\partial X_1}\right]$$

$$=\left(\frac{\partial \eta_2}{\partial X_0}+\frac{\partial \eta_1}{\partial X_1}\right)[E_0]\cos \omega_0 T_0 \quad \text{at} \quad y=0 \tag{2.31}$$

$$\left[\tilde{\varepsilon}\frac{\partial \psi_2}{\partial y}\right]=\frac{\partial \eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial \psi_1}{\partial X_0}\right]-\eta_1\left[\tilde{\varepsilon}\frac{\partial^2 \psi_1}{\partial y^2}\right] \quad \text{at} \quad y=0 \tag{2.32}$$

$$\frac{\partial\phi_2^{(1)}}{\partial T_0}-\rho\frac{\partial\phi_2^{(2)}}{\partial T_0}+(1-\rho)\eta_2$$

$$=-\eta_1\frac{\partial^2\phi_1^{(1)}}{\partial y\,\partial T_0}-\frac{\partial\phi_1^{(1)}}{\partial T_1}$$

$$+\rho\eta_1\frac{\partial^2\phi_1^{(2)}}{\partial y\,\partial T_0}+\rho\frac{\partial\phi_1^{(2)}}{\partial T_1}-\frac{1}{2}\left(\frac{\partial\phi_1^{(1)}}{\partial X_0}\right)^2+\frac{1}{2}\rho\left(\frac{\partial\phi_1^{(2)}}{\partial X_0}\right)^2$$

$$-\frac{1}{2}\left(\frac{\partial\phi_1^{(1)}}{\partial y}\right)^2+\frac{1}{2}\rho\left(\frac{\partial\phi_1^{(2)}}{\partial y}\right)^2+2\frac{\partial^2\eta_1}{\partial X_0\,\partial X_1}+\frac{\partial^2\eta_2}{\partial X_0^2}$$

$$-\frac{1}{2}\left[\tilde{\varepsilon}\left(\frac{\partial\psi_1}{\partial X_0}\right)^2\right]+\frac{1}{2}\left[\tilde{\varepsilon}\left(\frac{\partial\psi_1}{\partial y}\right)^2\right]-\left[\tilde{\varepsilon}E_0\frac{\partial\psi_2}{\partial y}\right]\cos\omega_0T_0$$

$$-\eta_1\left[\tilde{\varepsilon}E_0\frac{\partial^2\psi_1}{\partial y^2}\right]\cos\omega_0T_0-\left(\frac{\partial\eta_1}{\partial X_0}\right)^2[\tilde{\varepsilon}E_0^2]\cos^2\omega_0T_0$$

$$+2\frac{\partial\eta_1}{\partial X_0}\left[\tilde{\varepsilon}E_0\frac{\partial\psi_1}{\partial X_0}\right]\cos\omega_0T_0-\frac{\partial\eta_1}{\partial T_0}\frac{\partial\phi_1^{(1)}}{\partial y}$$

$$+\rho\frac{\partial\eta_1}{\partial T_0}\frac{\partial\phi_1^{(2)}}{\partial y}\qquad\text{at}\quad y=0\tag{2.33}$$

The solutions of these equations are

$$\eta_2(X_0,X_1,X_2,T_0,T_1,T_2)=\alpha(T_0,T_1,T_2,X_1,X_2)\,e^{2iKX_0}+\text{c.c.}\tag{2.34}$$

$$\phi_2^{(2)}(X_0,X_1,X_2,y,T_0,T_1,T_2)$$

$$=-\frac{1}{K}\left\{\frac{\partial D}{\partial T_1}+\frac{i}{K}(1+Ky)\frac{\partial^2 D}{\partial T_0\,\partial X_1}\right\}$$

$$\times e^{iKX_0-Ky}-D\frac{\partial D}{\partial T_0}e^{2iKX_0-2Ky}-\frac{1}{2K}\frac{\partial\alpha}{\partial T_0}e^{2iKX_0-2Ky}$$

$$+\text{c.c.}\tag{2.35}$$

$$\phi_2^{(1)}(X_0,X_1,X_2,y,T_0,T_1,T_2)$$

$$=\frac{1}{K}\left\{\frac{\partial D}{\partial T_1}+\frac{i}{K}(1-Ky)\frac{\partial^2 D}{\partial T_0\,\partial X_1}\right\}$$

$$\times e^{iKX_0+Ky}-D\frac{\partial D}{\partial T_0}e^{2iKX_0+2Ky}+\frac{1}{2K}\frac{\partial\alpha}{\partial T_0}$$

$$\times e^{2iKX_0+2Ky}+\text{c.c.}\tag{2.36}$$

$$\psi_2^{(2)}(X_0, X_1, X_2, y, T_0, T_1, T_2)$$
$$= -i\frac{\partial D}{\partial X_1} yE_0^{(2)} \cos \omega_0 T_0$$
$$\times e^{iKX_0-Ky} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} - \tfrac{1}{2}KD^2 \frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(2)} \cos \omega_0 T_0$$
$$\times e^{2iKX_0-2Ky} - \alpha \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(2)} \cos \omega_0 T_0\, e^{2iKX_0-2Ky}$$
$$- KD^2 \frac{\tilde{\varepsilon}^{(2)}\tilde{\varepsilon}^{(1)}-(\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(2)} \cos \omega_0 T_0\, e^{2iKX_0-2Ky} + \text{c.c.} \tag{2.37}$$

$$\psi_2^{(1)}(X_0, X_1, X_2, y, T_0, T_1, T_2)$$
$$= -i\frac{\partial D}{\partial X_1} y \frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}$$
$$\times E_0^{(1)} \cos \omega_0 T_0\, e^{iKX_0+Ky} + \left[-\tfrac{1}{2}KD^2 \frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \right.$$
$$\left. + \alpha \frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} - KD^2 \frac{(\tilde{\varepsilon}^{(2)})^2-\tilde{\varepsilon}^{(1)}\tilde{\varepsilon}^{(2)}}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \right]$$
$$\times E_0^{(1)} \cos \omega_0 T_0\, e^{2iKX_0+2Ky} + \text{c.c.} \tag{2.38}$$

Substituting from equations (2.34)-(2.37) and (2.38) into equations (2.33) and equating the coefficients of respective powers of ε to zero, we get

$$\frac{\partial^2 D}{\partial T_0 \partial T_1} - \frac{i}{2(1+\rho)} \left(1-\rho+3K^2-2K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \right.$$
$$\left. \times E_0^{(1)}E_0^{(2)} \cos^2 \omega_0 T_0 \right) \frac{\partial D}{\partial X_1} = 0 \tag{2.39}$$

$$\frac{\partial^2 \alpha}{\partial T_0^2} + \frac{2K}{1+\rho} \left(1-\rho+4K^2-2K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(1)}E_0^{(2)} \cos^2 \omega_0 T_0 \right) \alpha$$
$$= -\frac{4K^3}{1+\rho} D^2 \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)}E_0^{(2)} \cos^2 \omega_0 T_0 \tag{2.40}$$

The solution of equation (2.40) is lengthy and will not be included here (it is available from the authors on request).

Order ε^3:

$$\frac{\partial^2 \phi_3^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \phi_3^{(2),(1)}}{\partial y^2} = -2\frac{\partial^2 \phi_2^{(2),(1)}}{\partial X_0 \partial X_1} - 2\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0 \partial X_2} - \frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_1^2} \tag{2.41}$$

$$\frac{\partial^2 \psi_3^{(2),(1)}}{\partial X_0^2}+\frac{\partial^2 \psi_3^{(2),(1)}}{\partial y^2}=-2\frac{\partial^2 \psi_2^{(2),(1)}}{\partial X_0\,\partial X_1}-2\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_0\,\partial X_2}-\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_1^2} \tag{2.42}$$

$$\begin{aligned}
&\frac{\partial \eta_3}{\partial T_0}+\frac{\partial \eta_2}{\partial T_1}+\frac{\partial \eta_1}{\partial T_2}+\frac{\partial \eta_1}{\partial X_0}\frac{\partial \phi_2^{(2),(1)}}{\partial X_0}+\eta_1\frac{\partial \eta_1}{\partial X_0}\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0\,\partial y}\\
&\quad+\frac{\partial \eta_1}{\partial X_0}\frac{\partial \phi_1^{(2),(1)}}{\partial X_1}+\left(\frac{\partial \eta_1}{\partial X_0}\right)^2\frac{\partial \phi_1^{(2),(1)}}{\partial y}\\
&=\eta_2\frac{\partial \phi_1^{(2),(1)}}{\partial y^2}\\
&\quad+\tfrac{1}{2}\eta_1^2\frac{\partial^3 \phi_1^{(2),(1)}}{\partial y^3}+\eta_1\frac{\partial^2 \phi_2^{(2),(1)}}{\partial y^2}+\frac{\partial \phi_3^{(2),(1)}}{\partial y}-\frac{\partial \phi_1^{(2),(1)}}{\partial X_0}\frac{\partial \eta_1}{\partial X_1}\\
&\quad-\frac{\partial \eta_2}{\partial X_0}\frac{\partial \phi_1^{(2),(1)}}{\partial X_0}\qquad \text{at}\quad y=0
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&2\frac{\partial \eta_2}{\partial X_0}\left[\frac{\partial \psi_1}{\partial y}\right]+2\frac{\partial \eta_1}{\partial X_1}\left[\frac{\partial \psi_1}{\partial y}\right]+2\eta_1\frac{\partial \eta_1}{\partial X_0}\left[\frac{\partial^2 \psi_1}{\partial y^2}\right]\\
&\quad+2\frac{\partial \eta_1}{\partial X_0}\left[\frac{\partial \psi_2}{\partial y}\right]+\eta_2\left[\frac{\partial^2 \psi_1}{\partial y\,\partial X_0}\right]+\tfrac{1}{2}\eta_1^2\left[\frac{\partial^3 \psi_1}{\partial y^2\,\partial X_0}\right]\\
&\quad+\eta_1\left[\frac{\partial^2 \psi_2}{\partial y\,\partial X_0}\right]+\left[\frac{\partial \psi_3}{\partial X_0}\right]+\left[\frac{\partial \psi_2}{\partial X_1}\right]+\eta_1\left[\frac{\partial^2 \psi_1}{\partial y\,\partial X_1}\right]+\left[\frac{\partial \psi_1}{\partial X_2}\right]\\
&=\left(\frac{\partial \eta_3}{\partial X_0}+\frac{\partial \eta_2}{\partial X_1}+\frac{\partial \eta_1}{\partial X_2}\right)[E_0]\cos\omega_0 T_0\qquad \text{at}\quad y=0
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
&\frac{\partial \eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial \psi_2}{\partial X_0}\right]+\eta_1\frac{\partial \eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial^2 \psi_1}{\partial y\,\partial X_0}\right]+\frac{\partial \eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial \psi_1}{\partial X_1}\right]\\
&\quad+\left(\frac{\partial \eta_2}{\partial X_0}+\frac{\partial \eta_1}{\partial X_1}\right)\left[\tilde{\varepsilon}\frac{\partial \psi_1}{\partial X_0}\right]+\left(\frac{\partial \eta_1}{\partial X_0}\right)^2\left[\tilde{\varepsilon}\frac{\partial \psi_1}{\partial y}\right]\\
&\quad=\eta_2\left[\tilde{\varepsilon}\frac{\partial^2 \psi_1}{\partial y^2}\right]+\tfrac{1}{2}\eta_1^2\left[\tilde{\varepsilon}\frac{\partial^3 \psi_1}{\partial y^3}\right]+\eta_1\left[\tilde{\varepsilon}\frac{\partial^2 \psi_2}{\partial y^2}\right]\\
&\quad+\left[\tilde{\varepsilon}\frac{\partial \psi_3}{\partial y}\right]\qquad \text{at}\quad y=0
\end{aligned} \tag{2.45}$$

$$
\begin{aligned}
&\frac{\partial\phi_3^{(1)}}{\partial T_0}-\rho\frac{\partial\phi_3^{(2)}}{\partial T_0}+(1-\rho)\eta_3-\frac{\partial^2\eta_3}{\partial X_0^2}+\left[\tilde{\varepsilon}E_0\frac{\partial\psi_3}{\partial y}\right]\cos\omega_0T_0\\
&\quad=-\eta_2\frac{\partial^2\phi_1^{(1)}}{\partial y\,\partial T_0}-\frac{\partial\eta_2}{\partial T_0}\frac{\partial\phi_1^{(1)}}{\partial y}-\tfrac{1}{2}\eta_1^2\frac{\partial^3\phi_1^{(1)}}{\partial y^2\,\partial T_0}-\eta_1\frac{\partial\eta_1}{\partial T_0}\frac{\partial^2\phi_1^{(1)}}{\partial y^2}\\
&\qquad-\eta_1\frac{\partial^2\phi_2^{(1)}}{\partial y\,\partial T_0}-\frac{\partial\eta_1}{\partial T_0}\frac{\partial\phi_2^{(1)}}{\partial y}-\frac{\partial\phi_2^{(1)}}{\partial T_1}-\eta_1\frac{\partial^2\phi_1^{(1)}}{\partial y\,\partial T_1}-\frac{\partial\eta_1}{\partial T_1}\frac{\partial\phi_1^{(1)}}{\partial y}\\
&\qquad-\frac{\partial\phi_1^{(1)}}{\partial T_2}+\rho\eta_2\frac{\partial^2\phi_1^{(2)}}{\partial y\,\partial T_0}+\rho\frac{\partial\eta_2}{\partial T_0}\frac{\partial\phi_1^{(2)}}{\partial y}+\tfrac{1}{2}\rho\eta_1^2\frac{\partial^3\phi_1^{(2)}}{\partial y^2\,\partial T_0}\\
&\qquad+\rho\eta_1\frac{\partial\eta_1}{\partial T_0}\frac{\partial^2\phi_1^{(2)}}{\partial y^2}+\rho\eta_1\frac{\partial^2\phi_2^{(2)}}{\partial y\,\partial T_0}+\rho\frac{\partial\eta_1}{\partial T_0}\frac{\partial\phi_2^{(2)}}{\partial y}\\
&\qquad+\rho\frac{\partial\phi_2^{(2)}}{\partial T_1}+\rho\eta_1\frac{\partial^2\phi_1^{(2)}}{\partial y\,\partial T_1}+\rho\frac{\partial\eta_1}{\partial T_1}\frac{\partial\phi_1^{(2)}}{\partial y}+\rho\frac{\partial\phi_1^{(2)}}{\partial T_2}\\
&\qquad-\frac{\partial\phi_1^{(1)}}{\partial X_0}\frac{\partial\phi_2^{(1)}}{\partial X_0}-\frac{\partial\eta_1}{\partial X_0}\frac{\partial\phi_1^{(1)}}{\partial X_0}\frac{\partial\phi_1^{(1)}}{\partial y}-\eta_1\frac{\partial\phi_1^{(1)}}{\partial X_0}\frac{\partial^2\phi_1^{(1)}}{\partial y\,\partial X_0}\\
&\qquad-\frac{\partial\phi_1^{(1)}}{\partial X_0}\frac{\partial\phi_1^{(1)}}{\partial X_1}+\rho\frac{\partial\phi_1^{(2)}}{\partial X_0}\frac{\partial\phi_2^{(2)}}{\partial X_0}+\rho\frac{\partial\eta_1}{\partial X_0}\frac{\partial\phi_1^{(2)}}{\partial X_0}\frac{\partial\phi_1^{(2)}}{\partial y}\\
&\qquad+\rho\eta_1\frac{\partial\phi_1^{(2)}}{\partial X_0}\frac{\partial^2\phi_1^{(2)}}{\partial y\,\partial X_0}+\rho\frac{\partial\phi_1^{(2)}}{\partial X_0}\frac{\partial\phi_1^{(2)}}{\partial X_1}-\eta_1\frac{\partial\phi_1^{(1)}}{\partial y}\frac{\partial^2\phi_1^{(1)}}{\partial y^2}\\
&\qquad-\frac{\partial\phi_1^{(1)}}{\partial y}\frac{\partial\phi_2^{(1)}}{\partial y}+\rho\eta_1\frac{\partial\phi_1^{(2)}}{\partial y}\frac{\partial^2\phi_1^{(2)}}{\partial y^2}+\rho\frac{\partial\phi_1^{(2)}}{\partial y}\frac{\partial\phi_2^{(2)}}{\partial y}\\
&\qquad+\frac{\partial^2\eta_1}{\partial X_1^2}+2\frac{\partial^2\eta_2}{\partial X_0\,\partial X_1}+2\frac{\partial^2\eta_1}{\partial X_0\,\partial X_2}-\frac{3}{2}\frac{\partial^2\eta_1}{\partial X_0^2}\left(\frac{\partial\eta_1}{\partial X_0}\right)^2\\
&\qquad-\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial X_0}\frac{\partial\psi_2}{\partial X_0}\right]-\frac{\partial\eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial X_0}\frac{\partial\psi_1}{\partial y}\right]-\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial X_0}\frac{\partial\psi_1}{\partial X_1}\right]\\
&\qquad-\eta_1\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial X_0}\frac{\partial^2\psi_1}{\partial y\,\partial X_0}\right]+\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial y}\frac{\partial\psi_2}{\partial y}\right]+\eta_1\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial y}\frac{\partial^2\psi_1}{\partial y^2}\right]\\
&\qquad-\eta_2\left[\frac{\partial^2\psi_1}{\partial y^2}\tilde{\varepsilon}E_0\right]\cos\omega_0T_0-\tfrac{1}{2}\eta_1^2\left[\tilde{\varepsilon}E_0\frac{\partial^3\psi_1}{\partial y^3}\right]\cos\omega_0T_0\\
&\qquad-\eta_1\left[\tilde{\varepsilon}E_0\frac{\partial^2\psi_2}{\partial y^2}\right]\cos\omega_0T_0+2\frac{\partial\eta_1}{\partial X_0}\left(\frac{\partial\eta_2}{\partial X_0}+\frac{\partial\eta_1}{\partial X_1}\right)[\tilde{\varepsilon}E_0^2]\cos^2\omega_0T_0\\
&\qquad+4\left(\frac{\partial\eta_1}{\partial X_0}\right)^2\left[\tilde{\varepsilon}E_0\frac{\partial\psi_1}{\partial y}\right]\cos\omega_0T_0+2\eta_1\frac{\partial\eta_1}{\partial X_0}\left[\tilde{\varepsilon}E_0\frac{\partial^2\psi_1}{\partial y\,\partial X_0}\right]\cos\omega_0T_0
\end{aligned}
$$

$$+2\frac{\partial\eta_1}{\partial X_0}\left[\tilde{\varepsilon}E_0\frac{\partial\psi_2}{\partial X_0}\right]\cos\omega_0 T_0+2\left(\frac{\partial\eta_2}{\partial X_0}+\frac{\partial\eta_1}{\partial X_1}\right)$$
$$\times\left[\tilde{\varepsilon}E_0\frac{\partial\psi_1}{\partial X_0}\right]\cos\omega_0 T_0-2\frac{\partial\eta_1}{\partial X_0}\left[\tilde{\varepsilon}\frac{\partial\psi_1}{\partial X_0}\frac{\partial\psi_1}{\partial y}\right]+2\frac{\partial\eta_1}{\partial X_0}$$
$$\times\left[\tilde{\varepsilon}E_0\frac{\partial\psi_1}{\partial X_1}\right]\cos\omega_0 T_0 \qquad \text{at} \quad y=0 \tag{2.46}$$

The solutions of these equations are

$$\eta_3(X_0, X_1, X_2, T_0, T_1, T_2)=\tfrac{1}{2}K^2D^2(X_1, X_2, T_0, T_1, T_2)\, e^{iKX_0}+\text{c.c.} \tag{2.47}$$

$$\phi_3^{(2)}(X_0, X_1, X_2, y, T_0, T_1, T_2)$$
$$=\left[-\frac{1}{K}\frac{\partial D}{\partial T_2}-\bar{D}\frac{\partial\alpha}{\partial T_0}+\alpha\frac{\partial\bar{D}}{\partial T_0}\right.$$
$$-4KD\bar{D}\frac{\partial D}{\partial T_0}-\frac{i}{K^2}\frac{\partial^2 D}{\partial T_1\partial X_1}+\frac{1}{K^3}\frac{\partial^3 D}{\partial T_0\partial X_1^2}-\frac{i}{K^2}\frac{\partial^2 D}{\partial T_0\partial X_2}$$
$$\left.+\left(\frac{1}{K^2}\frac{\partial^3 D}{\partial T_0\partial X_1^2}-\frac{i}{K}\frac{\partial^2 D}{\partial T_1\partial X_1}-\frac{i}{K}\frac{\partial^2 D}{\partial T_0\partial X_2}\right)y+\frac{1}{2K}\frac{\partial^3 D}{\partial T_0\partial X_1^2}y^2\right]$$
$$\times\, e^{iKX_0-Ky}+\text{N.S.T.}+\text{c.c.} \tag{2.48}$$

$$\phi_3^{(1)}(X_0, X_1, X_2, y, T_0, T_1, T_2)$$
$$=\left[\frac{1}{K}\frac{\partial D}{\partial T_2}-\bar{D}\frac{\partial\alpha}{\partial T_0}+\alpha\frac{\partial\bar{D}}{\partial T_0}+4KD\bar{D}\frac{\partial D}{\partial T_0}\right.$$
$$+\frac{i}{K^2}\frac{\partial^2 D}{\partial T_1\partial X_1}-\frac{1}{K^3}\frac{\partial^3 D}{\partial T_0\partial X_1^2}+\frac{i}{K^2}\frac{\partial^2 D}{\partial T_0\partial X_2}$$
$$+\left(\frac{1}{K^2}\frac{\partial^3 D}{\partial T_0\partial X_1^2}-\frac{i}{K}\frac{\partial^2 D}{\partial T_1\partial X_1}-\frac{i}{K}\frac{\partial^2 D}{\partial T_0\partial X_2}\right)y$$
$$\left.-\frac{1}{2K}\frac{\partial^3 D}{\partial T_0\partial X_1^2}y^2\right]e^{iKX_0+Ky}+\text{N.S.T.}+\text{c.c.} \tag{2.49}$$

$$\psi_3^{(1)}(X_0, X_1, X_2, y, T_0, T_1, T_2)$$
$$=\left[-2K^2D^2\bar{D}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}\cos\omega_0 T_0\right.$$
$$+4K^2D^2\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3}\tilde{\varepsilon}^{(1)}E_0^{(1)}\cos\omega_0 T_0$$
$$-2K\alpha\bar{D}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\tilde{\varepsilon}^{(2)}E_0^{(1)}\cos\omega_0 T_0$$

$$
\begin{aligned}
&-K\alpha\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}\cos\omega_0T_0\\
&-i\frac{\partial D}{\partial X_2}y\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}\cos\omega_0T_0\\
&-\frac{1}{2}\frac{\partial^2D}{\partial X_1^2}y^2\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}\cos\omega_0T_0\Bigg]\\
&\times e^{iKX_0+Ky}+\text{N.S.T.}+\text{c.c.}
\end{aligned}
\tag{2.50}
$$

$$
\begin{aligned}
&\psi_3^{(2)}(X_0,X_1,X_2,y,T_0,T_1,T_2)\\
&=\Bigg[2K^2D^2\bar{D}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(2)}\cos\omega_0T_0\\
&+4K^2D^2\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3}\tilde{\varepsilon}^{(2)}E_0^{(2)}\cos\omega_0T_0\\
&+K\alpha\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(2)}\cos\omega_0T_0\\
&-2K\alpha\bar{D}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\tilde{\varepsilon}^{(1)}E_0^{(2)}\cos\omega_0T_0\\
&-i\frac{\partial D}{\partial X_2}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}yE_0^{(2)}\cos\omega_0T_0\\
&+\frac{1}{2}\frac{\partial^2D}{\partial X_1^2}y^2\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(2)}\cos\omega_0T_0\Bigg]e^{iKX_0-Ky}+\text{N.S.T.}+\text{c.c.}
\end{aligned}
\tag{2.51}
$$

Substituting from equations (2.47)-(2.51) and (2.19) into equation (2.46) and after some simplifications, we obtain the following differential equation:

$$
\begin{aligned}
&\frac{2}{K}(1+\rho)\frac{\partial^2D}{\partial T_0\,\partial T_2}+2(1-\rho)\alpha\frac{\partial^2\bar{D}}{\partial T_0^2}+3(1+\rho)KD\bar{D}\frac{\partial^2D}{\partial T_0^2}\\
&+4(1+\rho)K\bar{D}\left(\frac{\partial D}{\partial T_0}\right)^2+4(1+\rho)KD\frac{\partial D}{\partial T_0}\frac{\partial\bar{D}}{\partial T_0}\\
&+\frac{2i}{K^2}(1+\rho)\frac{\partial^3D}{\partial T_0\,\partial T_1\,\partial X_1}-\frac{1+\rho}{K^3}\frac{\partial^4D}{\partial T_0^2\,\partial X_1^2}\\
&+\frac{i}{K^2}(1+\rho)\frac{\partial^3D}{\partial T_0^2\,\partial X_2}-K^4D^2\bar{D}+\tfrac{1}{2}(1-\rho)K^2D^2\bar{D}\\
&+\tfrac{1}{2}(1+\rho)KD^2\frac{\partial^2\bar{D}}{\partial T_0^2}+\tfrac{11}{2}K^3D^2\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0
\end{aligned}
$$

$$-8K^3D^2\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3}\tilde{\varepsilon}^{(1)}\tilde{\varepsilon}^{(2)}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0$$

$$+i\frac{\partial D}{\partial X_2}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0+4(1-\rho)\frac{\partial\alpha}{\partial T_0}\frac{\partial\bar{D}}{\partial T_0}$$

$$+\frac{1+\rho}{K}\frac{\partial^2D}{\partial T_1^2}-\frac{\partial^2D}{\partial X_1^2}-2iK\frac{\partial D}{\partial X_2}+K^2\alpha\bar{D}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}$$

$$\times E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0-K^2\alpha\bar{D}(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0$$

$$+2K^2\alpha\bar{D}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\{(\tilde{\varepsilon}^{(1)})^2+(\tilde{\varepsilon}^{(2)})^2\}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0=0 \qquad (2.52)$$

Equations (2.52), (2.27), and (2.39) and the solution of equation (2.40) can be used to study the propagation of a finite-amplitude wave train over the surface. Using (2.27), (2.39), and the solution of equation (2.40) in equation (2.52) produces the system of equations

$$\frac{\partial^2A_0}{\partial x^2}+\beta_0(t)A_0^2\bar{A}_0=0 \qquad (2.53)$$

$$\frac{\partial^2A_1}{\partial x^2}+\beta_1(t)A_1=\beta_2(t)A_0^2\bar{A}_0 \qquad (2.54)$$

The solutions of these equations can be written in the form

$$A_0(x,t)=C_1\,e^{i\sqrt{\beta_0(t)}x} \qquad \text{where} \quad C_1=\pm1 \qquad (2.55)$$

$$A_1(x,t)=\frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)}\,e^{i\sqrt{\beta_0(t)}x} \qquad (2.56)$$

where $\beta_0(t)$, $\beta_1(t)$, $\beta_2(t)$, and a_0 are lengthy and will not be included here (they are available from the authors on request).

Substituting from equations (2.55) and (2.56) into equation (2.27), we obtain

$$D(x,t)=\sum\varepsilon_1^nA_n(x,t)\,e^{i(\sqrt{a}+2n)\omega_0t}$$

$$=C_1\,e^{i\sqrt{\beta_0(t)}x}\,e^{i\sqrt{a}\omega_0t}$$

$$+\varepsilon_1\frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)}\,e^{i(\sqrt{\beta_0(t)}x+(\sqrt{a}+2)\omega_0t)}+O(\varepsilon_1^2) \qquad (2.57)$$

Substituting for D and α into η, we finally obtain the following expression for the interface displacement η:

$$\eta(x, t) = \varepsilon \left(e^{i(\sqrt{\beta_0(t)}\,x + \sqrt{a}\,\omega_0 t)} + \varepsilon_1 \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+2)\omega_0 t)} \right) e^{iKx}$$
$$+ \varepsilon^2 \left(b_1\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+1)\omega_0 t)} \right.$$
$$+ b_2\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}-1)\omega_0 t)} + b_3\, e^{2i(\sqrt{\beta_0(t)}\,x + \sqrt{a}\,\omega_0 t)}$$
$$+ b_3\, e^{2i\omega_0 t} + b_4 + \varepsilon_1 \left\{ b_5\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+1)\omega_0 t)} \right.$$
$$+ b_6\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}-1)\omega_0 t)} + b_7\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+2)\omega_0 t)}$$
$$+ b_8\, e^{2i(\sqrt{\beta_0(t)}\,x + \sqrt{a}\,\omega_0 t)}$$
$$+ b_9 \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+1)\omega_0 t)}$$
$$+ b_{10} \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{2i(\sqrt{\beta_0(t)}\,x + \sqrt{a}\,\omega_0 t)}$$
$$+ b_{11} \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{2i(\sqrt{\beta_0(t)}\,x + (\sqrt{a}+2)\omega_0 t)}$$
$$+ b_{12} \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{2i(\sqrt{\beta_0(t)}\,x + \sqrt{a}\,\omega_0 t)} + b_{13}\, e^{2i\omega_0 t}$$
$$+ b_{14}\, e^{4i\omega_0 t} + b_{15} + b_{16} \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{2i\omega_0 t}$$
$$\left.\left. + b_{17} \frac{\beta_2(t)}{\beta_1(t) - \beta_0(t)}\, e^{4i\omega_0 t} \right\} \right) e^{2iKx} + O(\varepsilon^3) + \text{c.c.} \tag{2.58}$$

where the b's are constants and the details are lengthy and will not be included here (they are available from the authors on request).

For $\varepsilon\varepsilon_1 \ll 1$, we can write $\eta(x, t)$ as follows:

$$\eta(x, t) = \varepsilon\, e^{i[(\sqrt{\beta_0(t)}+K)x + \sqrt{a}\,\omega_0 t]} + \varepsilon^2 (b_1\, e^{2i[(\sqrt{\beta_0(t)}+K)x + (\sqrt{a}+1)\omega_0 t]}$$
$$+ b_2\, e^{2i[(\sqrt{\beta_0(t)}\,x + Kx) + (\sqrt{a}-1)\omega_0 t]}$$
$$+ b_3 \{ e^{2i[(\sqrt{\beta_0(t)}+K)x + \sqrt{a}\,\omega_0 t]} + e^{2i(Kx + \omega_0 t)} \}$$
$$+ b_4\, e^{2iKx}) + O(\varepsilon^3) + \text{c.c.} \tag{2.59}$$

3. STABILITY CONDITIONS

The analysis of this section will be based on equation (2.59). Equation (2.59) can be written as follows:

$$\begin{aligned}\eta(x, t) = {} & 2\varepsilon \cos[(\sqrt{\beta_0(t)} + K)x + \sqrt{a}\,\omega_0 t] \\ & + 2\varepsilon^2\{b_1 \cos 2[(\sqrt{\beta_0(t)} + K)x + (\sqrt{a} + 1)\omega_0 t] \\ & + b_2 \cos 2[(\sqrt{\beta_0(t)} + K)x + (\sqrt{a} - 1)\omega_0 t] \\ & + b_3 \cos 2[(\sqrt{\beta_0(t)} + K)x + \sqrt{a}\,\omega_0 t] \\ & + b_3 \cos 2(Kx + \omega_0 t) + b_4 \cos 2Kx\} \\ & + O(\varepsilon^3) + \text{c.c.}\end{aligned} \tag{3.1}$$

Thus, a finite-amplitude wave propagating through the surface is unstable when $a < 0$ and $\beta_0(t) < 0$, where

$$a = 1 - \rho + K^2 - \frac{K}{2}\frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \tag{3.2}$$

$$\beta_0(t) = \frac{f_1 - f_{11}(t) E_0^{(1)} E_0^{(2)} + f_{12}(E_0^{(1)})^2 (E_0^{(2)})^2 - f_{13}(E_0^{(1)})^3 (E_0^{(2)})^3}{f_{14} + f_{15}(t) E_0^{(1)} E_0^{(2)} + f_{16}(E_0^{(1)})^2 (E_0^{(2)})^2} \tag{3.3}$$

where the f's are evaluated; the details are lengthy and will not be included here (they are available from the authors on request).

The wave propagating through the surface is unstable when the conditions $a < 0$ and $\beta_0(t) < 0$ are satisfied, i.e.,

$$\left[E_0^{(1)} E_0^{(2)} < \frac{1 - \rho + K^2}{\alpha'_E} \qquad \text{where} \quad \alpha'_E = \frac{K}{2}\frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}}\right] \tag{3.4}$$

and

$$f_1 - f_{11}(t) E_0^{(1)} E_0^{(2)} + f_{12}(t)(E_0^{(1)})^2 (E_0^{(2)})^2 - f_{13}(t)(E_0^{(1)})^3 (E_0^{(2)})^3 > 0 \tag{3.5}$$

and

$$(E_0^{(1)})^2 (E_0^{(2)})^2 + \frac{f_{15}(t)}{f_{16}(t)} E_0^{(1)} E_0^{(2)} + \frac{f_{14}(t)}{f_{16}(t)} < 0 \tag{3.6}$$

The equality of relation (3.6) is a quadratic equation in $E_0^{(1)} E_0^{(2)}$. Inequality (3.6) is satisfied only if either $E_0^{(1)} E_0^{(2)} < E_2^2$ or $E_0^{(1)} E_0^{(2)} > E_3^2$. That is, either

$$E_0^{(1)} E_0^{(2)} < E_2^2 = \frac{1}{2}\left(\frac{-f_{15}}{f_{16}} + \left\{\left(\frac{f_{15}}{f_{16}}\right)^2 - 4\frac{f_{14}}{f_{16}}\right\}^{1/2}\right)$$

or (3.7)

$$E_0^{(1)} E_0^{(2)} > E_3^2 = \frac{1}{2}\left(\frac{-f_{15}}{f_{16}} - \left\{\left(\frac{f_{15}}{f_{16}}\right)^2 - 4\frac{f_{14}}{f_{16}}\right\}^{1/2}\right)$$

provided that $\{(f_{15}/f_{16})^2-4f_{14}/f_{16}\}^{1/2}$ is real. Or

$$E_0^{(1)}E_0^{(2)}<(1-\rho+K^2)/\alpha'_E$$

and

$$f_1-f_{11}(t)E_0^{(1)}E_0^{(2)}+f_{12}(E_0^{(1)})^2(E_0^{(2)})^2-f_{13}(E_0^{(1)})^3(E_0^{(2)})^3<0 \tag{3.8}$$

and

$$(E_0^{(1)})^2(E_0^{(2)})^2+\frac{f_{15}(t)}{f_{16}(t)}E_0^{(1)}E_0^{(2)}+\frac{f_{14}(t)}{f_{16}(t)}>0 \tag{3.9}$$

Inequality (3.9) is satisfied only if, $E_0^{(1)}E_0^{(2)}>E_2^2$ or $E_0^{(1)}E_0^{(2)}<E_3^2$. That is, either

$$E_0^{(1)}E_0^{(2)}>E_2^2=\frac{1}{2}\left(\frac{-f_{15}}{f_{16}}+\left\{\left(\frac{f_{15}}{f_{16}}\right)^2-4\frac{f_{14}}{f_{16}}\right\}^{1/2}\right)$$

or (3.10)

$$E_0^{(1)}E_0^{(2)}<E_3^2=\frac{1}{2}\left(\frac{-f_{15}}{f_{16}}-\left\{\left(\frac{f_{15}}{f_{16}}\right)^2-4\frac{f_{14}}{f_{16}}\right\}^{1/2}\right)$$

From the foregoing discussion, we see that the system is unstable provided that the electric field satisfies either of the following conditions:

$$E_0^{(1)}E_0^{(2)}<\frac{1-\rho+K^2}{\alpha'_E}$$

$$f_1-f_{11}(t)E_0^{(1)}E_0^{(2)}+f_{12}(t)(E_0^{(1)})^2(E_0^{(2)})^2-f_{13}(t)(E_0^{(1)})^3(E_0^{(2)})^3>0 \tag{3.11}$$

and

$$E_0^{(1)}E_0^{(2)}<E_2^{(2)} \qquad \text{or} \qquad E_0^{(1)}E_0^{(2)}>E_3^2$$

or

$$E_0^{(1)}E_0^{(2)}<\frac{1-\rho+K^2}{\alpha'_E}$$

$$f_1-f_{11}(t)E_0^{(1)}E_0^{(2)}+f_{12}(t)(E_0^{(1)})^2(E_0^{(2)})^2-f_{13}(t)(E_0^{(1)})^3(E_0^{(2)})^3<0$$

and (3.12)

$$E_0^{(1)}E_0^{(2)}>E_2^2 \qquad \text{or} \qquad E_0^{(1)}E_0^{(2)}<E_3^2$$

4. THE CASE OF A CONSTANT FIELD

A special case occurs when the applied electric field is constant, letting $\omega = 0$ in the boundary value equation (2.52). The condition becomes

$$\frac{\partial^2 A'(x)}{\partial x^2} + \gamma_0'(A')^2 A' = 0 \tag{4.1}$$

where

$$\begin{aligned}\gamma_0' = &\left\{ b\varepsilon^2 \left(\tfrac{7}{2}(1+\rho)K + \frac{12(1-\rho)K^2(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^3}{2K^2 - 1 + \rho(\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})} E_0^{(1)} E_0^{(2)} \right) \right. \\ &- \varepsilon^2 \tfrac{1}{2} K^2 \left(6(1-\rho) + 9K^2 - 18K \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \right) \\ &\left. + 8K^3 \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})^3} \tilde{\varepsilon}^{(2)} \tilde{\varepsilon}^{(1)} E_0^{(1)} E_0^{(2)} \right\} \\ &\times \left\{ \frac{1}{4K(1+\rho)} \left(1 - \rho + 3K^2 - 2K \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \right) \right. \\ &\left. + \left(3K - \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \right) \right\}^{-1}\end{aligned}$$

and

$$b = \frac{K}{1+\rho} \left(1 - \rho + K^2 - K \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \right)$$

The solution of this equation is

$$A'(x) = c\, e^{i\sqrt{\gamma_0'}x} \qquad \text{and} \qquad c = \pm 1 \tag{4.2}$$

where

$$D'(x, t) = A'(x)\, e^{i\sqrt{b\omega_0}t} + \text{c.c.} \tag{4.3}$$

and

$$\begin{aligned}\alpha'(x, t) = &-\frac{2K^2 E_0^{(1)} E_0^{(2)}}{2K^2 - 1 + \rho} e^{2i\sqrt{b\omega_0}t} (A')^2 \frac{(\tilde{\varepsilon}^{(2)}) - \tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})^2} - 4K^2 \frac{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})^2} \\ &\times \frac{A'\bar{A}'}{(1 - \rho + 4K^2 - 2K\{(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 / (\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})\} E_0^{(1)} E_0^{(2)})} \\ &+ \text{c.c.}\end{aligned} \tag{4.4}$$

Substituting for D' and α' into η', we finally obtain the following expression for the interface displacement η':

$$\eta'(x, t) = \varepsilon\, e^{i\sqrt{\gamma_0'}x+\sqrt{b}\omega_0 t)}\, e^{ikx}$$
$$+\varepsilon^2\Bigg(\frac{-2K^2E_0^{(1)}E_0^{(2)}}{2K^2-1+\rho}\,\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\, e^{2i(\sqrt{\gamma_0'}x+\sqrt{b}\omega_0 t)+2iKx}$$
$$-4K^2\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}$$
$$\times\frac{1}{(1-\rho+4K^2-2K\{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2/(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})\}E_0^{(1)}E_0^{(2)})}\Bigg)$$
$$+\tfrac{1}{2}\varepsilon^3K^2\, e^{i\sqrt{\gamma_0'}+K)x+i\sqrt{b}\omega_0 t}+\text{c.c.} \tag{4.5}$$

Equation (4.5) is similar to that obtained by Mohamed and El Shehawey (1983*a*).

4.1. Stability Conditions

The analysis of this section will be based on equation (4.5). Thus, a finite-amplitude wave propagating through the surface is unstable when $\gamma_0'<0$ and $b<0$, where

$$\gamma_0' = \frac{(E_0^{(1)})^2(E_0^{(2)})^2+F_{11}E_0^{(1)}E_0^{(2)}+F_{12}}{-F_{13}+E_0^{(1)}E_0^{(2)}} \tag{4.6}$$

and

$$b = \frac{K}{1+\rho}\left(1-\rho+K^2-K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}\, E_0^{(1)}E_0^{(2)}\right) \tag{4.7}$$

where the F's are evaluated; the details are lengthy and will not be included here (they are available from the authors on request).

The wave propagating through the surface is unstable when conditions $b<0$ and $\gamma_0'<0$ are satisfied, i.e.,

$$\left(E_0^{(1)}E_2^{(2)} < \frac{(1-\rho+K^2)}{\alpha_E^*},\qquad \text{where}\quad \alpha_E^* = K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}\right) \tag{4.8}$$

and

$$(E_0^{(1)})^2(E_0^{(2)})^2+F_{11}E_0^{(1)}E_0^{(2)}+F_{12}>0 \tag{4.9}$$

$$E_0^{(1)}E_0^{(2)}-F_{13}<0 \tag{4.10}$$

The equality of relation (4.9) is a quadratic equation in $E_0^{(1)}E_0^{(2)}$. Inequality (4.9) is satisfied only if either $E_0^{(1)}E_0^{(2)} < E_4^2$ or $E_0^{(1)}E_0^{(2)} > E_5^2$. That is, either

$$E_0^{(1)}E_0^{(2)} < E_4^2 = \tfrac{1}{2}[-F_{11} + (F_{11}^2 - 4F_{12})^{1/2}]$$

or (4.11)

$$E_0^{(1)}E_0^{(2)} > E_5^2 = \tfrac{1}{2}[-F_{11} - (F_{11}^2 - 4F_{12})^{1/2}]$$

provided that $(F_{11}^2 - 4F_{12})^{1/2}$ is real: or

$$E_0^{(1)}E_0^{(2)} > \frac{1-\rho+K^2}{\alpha_E^*}$$

and

$$(E_0^{(1)})^2(E_0^{(2)})^2 + F_{11}E_0^{(1)}E_0^{(2)} + F_{12} < 0 \tag{4.12}$$

$$E_0^{(1)}E_0^{(2)} - F_{13} > 0 \tag{4.13}$$

The equality of relation (4.12) is a quadratic equation in $E_0^{(1)}E_0^{(2)}$. Inequality (4.12) is satisfied only if $E_0^{(1)}E_0^{(2)} > E_4^2$ or $E_0^{(1)}E_0^{(2)} < E_5^2$. That is, either

$$E_0^{(1)}E_0^{(2)} > E_4^2 = \tfrac{1}{2}[-F_{11} + (F_{11}^2 - 4F_{12})^{1/2}]$$

or (4.14)

$$E_0^{(1)}E_0^{(2)} < E_5^2 = \tfrac{1}{2}[-F_{11} - (F_{11} - 4F_{12})^{1/2}]$$

From the foregoing discussion we see that the system is unstable provided that the electric field satisfies either of the following conditions:

$$E_0^{(1)}E_0^{(2)} > \frac{1-\rho+K^2}{\alpha_E^*}$$

$$E_0^{(1)}E_0^{(2)} < E_4^2 \qquad \text{or} \qquad E^{(1)}E_0^{(2)} > E_5^2 \tag{4.15}$$

and

$$E_0^{(1)}E_0^{(2)} < F_{13}$$

or

$$E_0^{(1)}E_0^{(2)} > \frac{1-\rho+K^2}{\alpha_E^*}$$

$$E_0^{(1)}E_0^{(2)} > E_4^2 \qquad \text{or} \quad E_0^{(1)}E_0^{(2)} < E_5^2 \tag{4.16}$$

and

$$E_0^{(1)}E_0^{(2)} > F_{13}$$

APPENDIX

If we substitute for η, ψ, ϕ as given by equations (2.34)-(2.37) and (2.38) into equation (2.33), we get

$$\frac{\partial^2\alpha}{\partial T_0^2}+\frac{2K}{1+\rho}\left(1-\rho+4K^2-2K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0\right)\alpha$$

$$=-\frac{4K^3}{1+\rho}D^2\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}E_0^{(2)}\cos^2\omega_0T_0 \tag{A1}$$

We assume a regular perturbation expansion for α,

$$\alpha=\alpha_0+\varepsilon_1\alpha_1+\varepsilon_1^2\alpha_2+O(\varepsilon_1^3) \tag{A2}$$

where $\varepsilon_1=q$.

Substituting α in equation (A2) into the nonhomogeneous Mathieu equation (A1) and comparing powers of ε_1 gives a sequence of equations, and substituting D in equation (2.27), after solving to all orders in perturbation theory, gives the following.

Equate coefficients of ε_1^0:

$$\frac{\partial^2\alpha_0}{\partial\xi^2}+a_0\alpha_0=\frac{-2K^3}{\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}E_0^{(2)}(1+\cos 2\xi)$$
$$\times(A_0^2\,e^{2i\sqrt{a}\omega_0T_0}+2A_0\bar{A})+\text{c.c.}$$

with solution

$$\alpha_0=\frac{-2K^3}{\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}E_0^{(2)}$$
$$\times\left(\frac{A_0^2}{a_0-4a}e^{2i\sqrt{a}\omega_0T_0}+\frac{2}{a_0}A_0\bar{A}+\frac{A_0^2\,e^{2i(\sqrt{a}+1)\omega_0T_0}}{2[a_0-4(\sqrt{a}+1)^2]}\right.$$
$$\left.+\frac{A_0^2\,e^{2i(\sqrt{a}-1)\omega_0T_0}}{2[a_0-4(\sqrt{a}-1)^2]}+\frac{A_0\bar{A}_0}{a_0-4}e^{2i\omega_0T_0}\right)+\text{c.c.} \tag{A3}$$

where

$$\varepsilon_1=\frac{K^2}{4\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}$$

$$a_0=\frac{2K}{\omega_0^2(1+\rho)}\left(1-\rho+4K^2-K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\right)$$

$$\xi=\omega_0T_0$$

Equate coefficients of ε_1:

$$\frac{\partial^2\alpha_1}{\partial\xi^2}+a_0\alpha_1=8\alpha_0\cos 2\xi-\frac{2K^3}{\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}E_0^{(2)}.$$
$$\times(1+\cos 2\xi)(2A_0A_1\,e^{2i(\sqrt{a}+1)\xi}+2\bar{A}_0A_1\,e^{2i\xi})+\text{c.c.}$$

with solution

$$\alpha_1=\frac{-2K^3}{\omega_0^2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}E_0^{(1)}E_0^{(2)}$$
$$\times\left(\frac{4A_0^2}{(a_0-4a)[a_0-4(\sqrt{a}+1)^2]}\,e^{2i(\sqrt{a}+1)\omega_0T_0}\right.$$
$$+\frac{4A_0^2}{(a_0-4a)[a-4(\sqrt{a}-1)^2]}\,e^{2i(\sqrt{a}-1)\omega_0T_0}$$
$$+\frac{8}{a_0(a_0-4)}A_0\bar{A}_0\,e^{2i\omega_0T_0}+\frac{2A_0^2\,e^{2i(\sqrt{a}+2)\omega_0T_0}}{[a_0-4(\sqrt{a}+1)^2][a_0-4(\sqrt{a}+2)^2]}$$
$$+\frac{2A_0^2\,e^{2i\sqrt{a}\omega_0T_0}}{[a_0-4(\sqrt{a}+1)^2](a_0-4a)}+\frac{2A_0^2\,e^{2i\sqrt{a}\omega_0T_0}}{[a_0-4(\sqrt{a}-1)^2](a_0-4a)}$$
$$+\frac{2A_0^2\,e^{2i(\sqrt{a}-2)\omega_0T_0}}{[a_0-4(\sqrt{a}-1)^2][a_0-4(\sqrt{a}-2)^2]}+\frac{4A_0\bar{A}_0\,e^{4i\omega_0T_0}}{(a_0-4)(a_0-16)}$$
$$+\frac{4A_0\bar{A}_0}{a_0(a_0-4)}+\frac{2A_0A_1\,e^{2i(\sqrt{a}+1)\omega_0T_0}}{[a_0-4(\sqrt{a}+1)^2]}+\frac{2\bar{A}_0A_1}{a_0-4}\,e^{2i\omega_0T_0}$$
$$\left.+\frac{A_0A_1}{a_0-4a}\,e^{2i\sqrt{a}\omega_0T_0}+\frac{A_0A_1\,e^{2i(\sqrt{a}+2)\omega_0T_0}}{a_0-4(\sqrt{a}+2)^2}+\frac{A_0A_1}{a_0}\right)+\text{c.c.} \tag{A4}$$

Substituting equations (A3), (A4), (2.55), and (2.56) into equation (A2) and putting $X_n=\varepsilon^nx$, $T_n=\varepsilon^nt$, we get

$$\alpha(x,t)=b_1\,e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}+1)\omega_0t)}+b_2\,e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}-1)\omega_0t}$$
$$+b_3\,e^{2i(\sqrt{\beta_0(t)}x+\sqrt{a}\omega_0t)}+b_3\,e^{2i\omega_0t}+b_4$$
$$+\varepsilon_1\left(b_5\,e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}+1)\omega_0t)}\right.$$
$$+b_6\,e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}-1)\omega_0t)}$$
$$+b_7\,e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}+2)\omega_0t)}$$
$$+b_8\,e^{2i(\sqrt{\beta_0(t)}x+\sqrt{a}\omega_0t)}$$
$$+b_9\frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)}\,e^{2i(\sqrt{\beta_0(t)}x+\sqrt{a}+1)\omega_0t)}$$

$$+ b_{10} \frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)} e^{2i(\sqrt{\beta_0(t)}x+\sqrt{a}\omega_0 t)}$$

$$+ b_{11} \frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)} e^{2i(\sqrt{\beta_0(t)}x+(\sqrt{a}+2)\omega_0 t)}$$

$$+ b_{12} \frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)} e^{2i(\sqrt{\beta_0(t)}x+\sqrt{a}\omega_0 t)} + b_{13} e^{2i\omega_0 t}$$

$$+ b_{14} e^{4i\omega_0 t} + b_{15} + b_{16} \frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)} e^{2i\omega_0 t}$$

$$+ b_{17} \frac{\beta_2(t)}{\beta_1(t)-\beta_0(t)} e^{4i\omega_0 t}\Bigg) + O(\varepsilon_1^2) + \text{c.c.} \qquad \text{(A5)}$$

where

$$b_1 = -\frac{K^3}{\omega_0^2(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \frac{E_0^{(1)}E_0^{(2)}}{a_0-4(\sqrt{a}+1)^2} \qquad \text{(A6)}$$

$$b_2 = \frac{-K^3}{\omega_0^2(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \frac{E_0^{(1)}E_0^{(2)}}{a_0-4(\sqrt{a}-1)^2} \qquad \text{(A7)}$$

$$b_3 = \frac{-2K^3}{\omega_0^2(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \frac{E_0^{(1)}E_0^{(2)}}{a_0-4a} \qquad \text{(A8)}$$

$$b_4 = \frac{-4K^3}{a_0\omega_0^2(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)}E_0^{(2)} \qquad \text{(A9)}$$

$$b_5 = \frac{-16K^3E_0^{(1)}E_0^{(2)}}{\omega_0^2(1+\rho)(a_0-4a)[a_0-4(\sqrt{a}+1)^2]} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \qquad \text{(A10)}$$

$$b_6 = \frac{-16K^3E_0^{(1)}E_0^{(2)}}{\omega_0^2(1+\rho)(a_0-4a)[a_0-4(\sqrt{a}-1)^2]} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \qquad \text{(A11)}$$

$$b_7 = \frac{-8K^3E_0^{(1)}E_0^{(2)}}{\omega_0^2(1+\rho)[a_0-4(\sqrt{a}+1)^2][a_0-4(\sqrt{a}+2)^2]} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \qquad \text{(A12)}$$

$$b_8 = \frac{-8K^3E_0^{(1)}E_0^{(2)}}{\omega_0^2(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \Bigg(\frac{1}{(a_0-4a)[a_0-4(\sqrt{a}+1)^2]}$$

$$+ \frac{1}{(a_0+4a)[a_0-4(\sqrt{a}-1)^2]}$$

$$+ \frac{1}{[a_0-4(\sqrt{a}-1)^2][a_0-4(\sqrt{a}-2)^2]}\Bigg) \qquad \text{(A13)}$$

$$b_9 = \frac{-4K^3}{\omega_0^2(1+\rho)[a_0-4(\sqrt{a}+1)^2]} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A14}$$

$$b_{10} = \frac{-2K^3}{\omega_0^2(1+\rho)(a_0-4a)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A15}$$

$$b_{11} = \frac{-2K^3}{\omega_0^2(1+\rho)[a_0-4(\sqrt{a}+2)^2]} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A16}$$

$$b_{12} = \frac{-2K^3}{\omega_0^2 a_0(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A17}$$

$$b_{13} = \frac{-32K^3}{\omega_0^2 a_0(a_0-4)(1+\rho)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A18}$$

$$b_{14} = \frac{-16K^3}{\omega_0^2(1+\rho)(a_0-4)(a_0-16)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A19}$$

$$b_{15} = \frac{-16K^3}{\omega_0^2 a_0(1+\rho)(a_0-4)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A20}$$

$$b_{16} = \frac{-4K^3}{\omega_0^2(1+\rho)(a_0-4)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A21}$$

$$b_{17} = \frac{-4K^3}{\omega_0^2(1+\rho)(a_0-16)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \tag{A22}$$

$$\begin{aligned}
\beta_0(t) = \varepsilon^2 \Bigg\{ & K^2 \Bigg[3(1-\rho) + \tfrac{9}{2} K^2 - 9K \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \cos^2 \omega_0 t \\
& + 8K \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3} \tilde{\varepsilon}^{(1)} \tilde{\varepsilon}^{(2)} E_0^{(1)} E_0^{(2)} \cos^2 \omega_0 t \Bigg] \\
& + \frac{16K^3(1-\rho)a}{(1+\rho)(a_0-4a)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \\
& + \frac{4K^5}{\omega_0^2(1+\rho)(a_0-4a)} \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^6}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^4} (E_0^{(1)})^2 (E_0^{(2)})^2 \cos^2 \omega_0 t \\
& - \frac{4K^4(1-\rho)}{(1+\rho)^2(a_0-4a)\omega_0^2} \Bigg[1-\rho+K^2-K \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \\
& \times \cos^2 \omega_0 t \Bigg] \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} E_0^{(1)} E_0^{(2)} \Bigg\} \\
& \times \Bigg\{ \frac{1}{K} \Bigg[3K - \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \cos^2 \omega_0 t \Bigg] - \frac{1}{4(1+\rho)Ka\omega_0^2} \\
& \times \Bigg[1-\rho+3K^2-2K \frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} E_0^{(1)} E_0^{(2)} \cos^2 \omega_0 t \Bigg]^2 \Bigg\}^{-1}
\end{aligned} \tag{A23}$$

$$\beta_1(t)=\varepsilon^2\left(\frac{16(\sqrt{a}+1)(1-\rho)\sqrt{a}}{(1+\rho)[a_0-4(\sqrt{a}+1)^2]}\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^3}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^2}E_0^{(1)}E_0^{(2)}\right.$$
$$+\frac{8K^5}{\omega_0^2(1+\rho)}\frac{E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t}{(a_0-4(\sqrt{a}+1)^2)}\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^6}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^4}$$
$$-\frac{8K^4(1-\rho)}{\omega_0^2(1+\rho)^2}\frac{E_0^{(1)}E_0^{(2)}}{a_0-4(\sqrt{a}+1)^2}\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^3}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^2}$$
$$\left.\times\left[1-\rho+K^2-K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right]\right)$$
$$\times\left\{\frac{1}{K}\left[3K-\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right]\right.$$
$$-\frac{1}{4K(1+\rho)(2+\sqrt{a})^2\omega_0^2}\left[1-\rho+3K^2-2K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}\right.$$
$$\left.\left.\times E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right]^2\right\}^{-1} \tag{A24}$$

$$\beta_2(t)=\varepsilon^2\left\{-K^2\left[6(1-\rho)+9K^2-18K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right.\right.$$
$$\left.+8K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^3}\tilde\varepsilon^{(1)}\tilde\varepsilon^{(2)}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right]$$
$$+\frac{32(1-\rho)K}{(1+\rho)(a_0-4a)[a_0-4(\sqrt{a}+1)^2]}\frac{K^3}{\omega_0^2(1+\rho)}\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^3}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^2}E_0^{(1)}E_0^{(2)}$$
$$\times\left[1-\rho+K^2-K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right]$$
$$-\frac{64K^3(1-\rho)2\sqrt{a}(\sqrt{a}+1)}{(1+\rho)(a_0-4a)[a_0-4(\sqrt{a}+1)^2]}\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^3}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^2}E_0^{(1)}E_0^{(2)}$$
$$-\frac{32K^5}{\omega_0^2(1+\rho)(a_0-4a)[a_0-4(\sqrt{a}+1)^2]}$$
$$\left.\times\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^6}{(\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)})^4}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right\}$$
$$\times\left\{\frac{1}{K}\left[3K-\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}\right.\right.$$
$$\times E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t-\frac{1}{4(1+\rho)(2+\sqrt{a})^2\omega_0^2}$$
$$\left.\left.\times\left(1-\rho+3K^2-2K\frac{(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{\tilde\varepsilon^{(1)}+\tilde\varepsilon^{(2)}}E_0^{(1)}E_0^{(2)}\cos^2\omega_0 t\right)^2\right]\right\}^{-1} \tag{A25}$$

$$a_0=\frac{2K}{\omega_0^2(1-\rho)}\left[1-\rho+4K^2-K\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}E_0^{(1)}E_0^{(2)}\right] \tag{A26}$$

$$f_1=4K(1-\rho+K^2)[3(1-\rho)+\tfrac{9}{2}K^2]+\frac{16K^5(1-\rho)}{(2K^2-1+\rho)(1+\rho)} \times(1-2\rho+2K^2+K^4+\rho^2-2\rho K^2) \tag{A27}$$

$$\begin{aligned} f_{11}={}&\tfrac{3}{2}K[(1-\rho)\tfrac{3}{2}-K^2]\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \\ &+\frac{16K^6(1-\rho)(1-\rho+K^2)}{(1+\rho)(2K^2-1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \\ &+8K^5\frac{(1-\rho)(1-\rho+K^2)}{(1+\rho)(2K^2-1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2} \\ &+4K^5(1-\rho+K^2)[9(\tilde{\varepsilon}^{(1)})^2+10\tilde{\varepsilon}^{(1)}\tilde{\varepsilon}^{(2)}+9(\tilde{\varepsilon}^{(2)})^2] \\ &\times\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})}\cos^2\omega_0 t \end{aligned} \tag{A28}$$

$$\begin{aligned} f_{12}(t)={}&\frac{4K^7(1-\rho)}{(1+\rho)(2K^2-1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^4}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}+4\frac{K^6(1-\rho)}{(1+\rho)(2K^2-1+\rho)} \\ &\times(1-\rho+K^2)\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^5}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3}+2K^6[9(\tilde{\varepsilon}^{(1)})^2+10\tilde{\varepsilon}^{(1)}\tilde{\varepsilon}^{(2)}+9(\tilde{\varepsilon}^{(2)})^2] \\ &\times\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^4}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^4}\cos^2\omega_0 t+\frac{8K^6(1-\rho+K^2)}{2K^2-1+\rho}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^6}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^4} \\ &\times\cos^2\omega_0 t+\frac{8K^6(1-\rho)}{(1+\rho)(2K^2-1+\rho)}(1-\rho+K^2)\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^5}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3} \\ &\times\cos^2\omega_0 t \end{aligned} \tag{A29}$$

$$\begin{aligned} f_{13}={}&\frac{4K^7}{2K^2-1+\rho}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^7}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})}\cos^2\omega_0 t \\ &+\frac{4K^7(1-\rho)}{(2K^2-1+\rho)(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^7}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^4}\cos^2\omega_0 t \end{aligned} \tag{A30}$$

$$f_{14}=3K^2[2(1-\rho)+K^2-(1-\rho)^2] \tag{A31}$$

$$f_{15}=K^3(4\cos^2\omega_0 t-3)\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}} \tag{A32}$$

$$f_{16}=-2K^2\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^4}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\cos^4\omega_0 t \tag{A33}$$

$$F_{11}=\left(\frac{12(1-\rho)K^2}{2K^2-1+\rho}\frac{\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)}}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}(1-\rho+K^2)-\tfrac{7}{2}(1+\rho)K\right.$$
$$\left.+9K^3+8K^3\frac{\tilde{\varepsilon}^{(1)}\tilde{\varepsilon}^{(2)}}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\right)\left(\frac{12K^2(\rho-1)}{2K^2-1+\rho}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^3}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^2}\right)^{-1} \tag{A34}$$

$$F_{12}=(\tfrac{7}{2}(1+\rho)(1-\rho+K^2)-3(1-\rho)K^2-\tfrac{9}{2}K^4)$$
$$\times\left(\frac{12(\rho-1)K^2}{2K^2-1+\rho}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^5}{(\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)})^3}\right)^{-1} \tag{A35}$$

$$F_{13}=\left(\frac{1}{4K(1+\rho)}(1-\rho+3K^2)+3K\right)$$
$$\times\left(\frac{1}{2(1+\rho)}\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}-\frac{(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\tilde{\varepsilon}^{(1)}+\tilde{\varepsilon}^{(2)}}\right)^{-1} \tag{A36}$$

REFERENCES

Bender, C., and Orszag, S. A. (1978). *Advanced Mathematical Methods for Scientists and Engineers.*

Berger, E., and Wille, R. (1972). Periodic flow phenomena, *Annual Review of Fluid Mechanics,* **4**, 313–329.

Hasimoto, H., and Ono, H. (1972). Nonlinear modulation of gravity waves, *Journal of the Physical Society of Japan,* **33**, 805.

Kant, R., Jindia, R. K., and Malik, S. K. (1981). *Quarterly of Applied Mathematics,* **1**, 23.

Melcher, J. R. (1963). *Field Coupled Surface Waves,* MIT Press, Cambridge, Massachusetts.

Michael, D. H. (1977). *Quarterly of Applied Mathematics,* **35**, 345.

Mohamed, A. A., and El Shehawey, E. F. (1983*a*). Nonlinear electrohydrodynamic Rayleigh–Taylor instability I: A perpendicular field in the absence of surface charges, *Journal of Fluid Mechanics,* **129**, 473–499.

Mohamed, A. A., and El Shehawey, E. F. (1983*b*). Nonlinear electrohydroynamic Rayleigh–Taylor instability. II: A perpendicular field producing surface charge, *Physics of Fluids,* **26**, 1724.

Mohamed, A. A., and El Shehawey, E. F. (1984). Nonlinear electrohydrodynamic Rayleigh–Taylor instability. III: Effect of a tangential field, *AJSE,* **9**(4), 345–360.

Nayfeh, A. H. (1973). *Perturbation Methods* (Wiley, New York, Chapter 6).

Nayfeh, A. H. (1977). Perturbation methods and nonlinear hyperbolic waves, *Journal of Sound Vibration* **54**, 605.

Nayfeh, A. H. (1976). *Transactions ASMEE Journal of Applied Mechanics,* **43**, 584.

Nayfeh, A. H., and Mook, D. T. (1977). Parametric excitations of linear systems having many degrees of freedom, *Journal of the Acoustical Society of America,* **62**, 375.

Nayfeh, A. H., and Mook, D. T. (1979). *Nonlinear Oscillations,* (Wiley, New York, pp. 290–338).

Rouch, N., and Mawhin, J. (1978). *Ordinary Differential Equations Stability and Periodic Solutions,* Boston.